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GENERALIZED RENEWAL SEQUENCES AND INFINITELY DIVISIBLE LATTICE DISTRIBUTIONS

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We introduce an increasing set of classes Γ_α $(0 \leq \alpha \leq 1)$ of infinitely divisible (i.d.) distributions on $\{0, 1, 2, \ldots\}$, such that Γ_0 is the set of all compound-geometric distributions and Γ_1 the set of all compound-Poisson distributions, i.e. the set of *all* i.d. distributions on the non-negative integers. These classes are defined by recursion relations similar to those introduced by Katti [4] for Γ_1 and by Steutel [7] for Γ_0. These relations can be regarded as generalizations of those defining the so-called renewal sequences (cf. [5] and [2]). Several properties of i.d. distributions now appear as special cases of properties of the Γ_α.

Infinite divisibility   Lattice distributions   Renewal sequences.

1. Introduction, summary and conventions

In [4] Katti gives the following theorem (see [8] for a simple proof).

Theorem 1.1. *A distribution $\{p_n\}_0^\infty$ on the non-negative integers, with $p_0 > 0$, is i.d. (i.e. compound-Poisson) iff there exist $r_n \geq 0$ $(n = 0, 1, 2, \ldots)$ such that*

$$(n+1)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k} \quad (n = 0, 1, 2, \ldots); \tag{1.1}$$

the r_n then satisfy $\sum_0^\infty r_n/(n+1) < \infty$.

In [7] the following similar theorem is proved.

Theorem 1.2. *A distribution $\{p_n\}_0^\infty$ on the non-negative integers, with $p_0 > 0$, is compound-geometric iff there exist $r_n' \geq 0$ $(n = 0, 1, 2, \ldots)$ such that*

$$p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}' \quad (n = 0, 1, 2, \ldots); \tag{1.2}$$

the r_n' then satisfy $\sum_0^\infty r_n' < 1$.

In fact, for any given distribution $\{p_n\}$ the equations (1.1) and (1.2) can be uniquely solved for r_n and r_n' (by Cramer's rule) to obtain explicit criteria for $\{p_n\}$ to be compound-Poisson, that is i.d., or compound-geometric. Conversely, any sequence of non-negative r_n or r'_n, satisfying $\sum r_n/(n \quad \infty$ or $\sum r'_n < 1$, by (1.1) or (1.2) uniquely determines a distribution $\{p_n\}$ that is compound-Poisson or compound-geometric. The equations (1.2) have the same structure as the equations defining the so-called renewal sequences (cf. [5]).

In this paper we consider a set of classes of i.d. distributions, that contains the classes defined by the Theorems 1.1 and 1.2 as special cases. For $0 \le \alpha \le 1$ we define the class Γ_α as follows.

Definition 1.3. A distribution $\{p_n\}_0^\infty$ on the non-negative integers, with $p_0 > 0$, belongs to Γ_α iff there exist $r_n(\alpha) \ge 0$ $(n = 0, 1, 2, \ldots)$ such that

$$c_n(\alpha) p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(\alpha) \quad (n = 0, 1, 2, \ldots); \tag{1.3}$$

the $r_n(\alpha)$ then satisfy

$$\sum_0^\infty r_n(\alpha)/c_n(\alpha) < \infty. \tag{1.4}$$

The quantities $c_n(\alpha)$, which will be specified in Section 2, are supposed to have the following properties.

$$\begin{aligned} &c_n(0) = 1, \quad c_n(1) = n + 1 \quad (n = 0, 1, 2, \ldots) \\ &c_n(\alpha) \text{ is non-decreasing in both } n \text{ and } \alpha. \end{aligned} \tag{1.5}$$

Remark. The concluding statement of Definition 1.3 (see also Theorem 1.1) can be proved as follows:

$$1 - p_0 = \sum_{n=0}^\infty p_{n+1} = \sum_{k=0}^\infty p_k \sum_{n=k}^\infty r_{n-k}(\alpha)/c_n(\alpha) \ge p_0 \sum_{n=0}^\infty r_n(\alpha)/c_n(\alpha).$$

If $c(\alpha) := \lim_{n\to\infty} c_n(\alpha)$ is finite, then (see also Theorem 1.2)

$$\sum_0^\infty r_n(\alpha) < c(\alpha). \tag{1.6}$$

For given $c_n(\alpha)$ the sequences $\{p_n\}$ and $\{r_n(\alpha)\}$, satisfying (1.4) or (1.6), determine each other uniquely by means of (1.3). Clearly, Γ_0 is the class of compound-geometric distributions and Γ_1 the class of all i.d. distributions. It is well-known (see e.g. [6, p. 320] and [2]) that

$$\Gamma_0 \subset \Gamma_1. \tag{1.7}$$

In order to obtain a classification of the i.d. distributions on the non-negative integers, we wish to choose the $c_n(\alpha)$ such that

$$\Gamma_\alpha \subset \Gamma_\beta \quad \text{if } \alpha < \beta \quad (0 \le \alpha < 1;\ 0 < \beta \le 1), \tag{1.8}$$

and we shall say that in this case the $c_n(\alpha)$ satisfy (1.8). The main result of this paper is that this can be done, and, in fact, that the choice

$$c_n(\alpha)=\frac{1-\alpha^{n+1}}{1-\alpha} \quad (0\leq \alpha \leq 1;\ n=0,1,2,\ldots), \tag{1.9}$$

with, of course, $c_n(1)=n+1$, produces classes Γ_α satisfying (1.8). We prove several properties of the Γ_α, some of which are generalizations of well-known properties of i.d. distributions. We exhibit examples of distributions $\{p_n\}$ such that $\{p_n\}\in\Gamma_\alpha$ for $\alpha\geq\alpha_0$ and $\{p_n\}\notin\Gamma_\alpha$ for $\alpha<\alpha_0$, and we prove that $\bigcup_{\alpha<1}\Gamma_\alpha$ is dense in Γ_1.

We shall frequently use (probability) generating functions ((p.)g.f.'s), and we adopt the following notational conventions. The g.f.'s of $\{a_n\}$, $\{b_n\},\ldots$ are denoted by $A(z), B(z),\ldots$, those of $\{r_n(\alpha)\}$ and $\{c_n(\alpha)\}$ by $R_\alpha(z)$ and $C_\alpha(z)$. Clearly, $P(z)$ and $C_\alpha(z)$ converge for $|z|<1$, whereas $R_\alpha(z)$ has a positive radius of convergence for every choice of the p_n, and is convergent for $|z|<1$ if $\{p_n\}\in\Gamma_\alpha$, which will also be denoted by $P\in\Gamma_\alpha$.

As we consider only distributions on the non-negative integers with positive mass at 0, we use the term distribution in that sense.

For full proofs, details, examples and counterexamples we refer to [3]. For general information on i.d. distributions we refer to [1, chapter XII] and [6, Chapter 5].

2. The choice of $c_n(\alpha)$

Clearly, there are many sequences $\{c_n(\alpha)\}$ satisfying (1.5). The most obvious candidates, however, do not satisfy (1.8). We mention four of those, which, to avoid confusion, we denote by $c_n^*(\alpha)$. The following simple choices violate (1.8), as may be seen from counterexamples (cf. [3]).

$$c_n^*(\alpha)=1+\alpha n,$$

$$c_n^*(\alpha)=(n+1)^\alpha,$$

$$c_n^*(\alpha)=(1+\alpha n)^\alpha.$$

Rather more sophisticated, one may try to interpolate between the g.f.'s $C_0(z)=(1-z)^{-1}$ and $C_1(z)=(1-z)^{-2}$, e.g. by $C_\alpha^*(z)=(1-z)^{-1-\alpha}$. This yields

$$c_n^*(\alpha)=\binom{\alpha+n}{n},$$

which somehow seems an attractive choice, but, unfortunately, also violates (1.8). We finally tried

$$C_\alpha(z)=(1-z)^{-1}(1-\alpha z)^{-1}, \tag{2.1}$$

corresponding to

$$c_n(\alpha)=\frac{1-\alpha^{n+1}}{1-\alpha} \quad (0\leq \alpha \leq 1;\ n=0,1,2,\ldots), \tag{2.2}$$

with $c_n(1)=n+1$. We shall prove in Section 3 that the $c_n(\alpha)$ given by (2.2) satisfy (1.8) (we note in passing that $\{c_n^*(\alpha)\}$ with $C_\alpha^*(z)=(1-z)^{-1}(1-\alpha z)^{-\alpha}$ does not).

The classes Γ_α as defined by (1.3) are now determined by the condition that the $r_n(\alpha)$, for a given distribution $\{p_n\}$ defined by

$$\frac{1-\alpha^{n+1}}{1-\alpha}p_{n+1}=\sum_{k=0}^{n}p_k r_{n-k}(\alpha), \tag{2.3}$$

should be non-negative, in which case $\sum_0^\infty r_n(\alpha)<(1-\alpha)^{-1}$ if $\alpha<1$ (cf. (1.6)). Equivalently, taking g.f.'s, R_α determined by

$$\frac{P(z)-P(\alpha z)}{z(1-\alpha)}=P(z)R_\alpha(z) \quad (0\leqslant\alpha<1), \tag{2.4}$$

should be absolutely monotone (a.m.). We state this as a lemma.

Lemma 2.1. *$P\in\Gamma_\alpha$ iff R_α is a.m., i.e. iff $1-P(\alpha z)/P(z)$ is a.m. $(0\leqslant\alpha<1)$.*

If $\alpha=0$ or $\alpha\uparrow 1$, we can solve (2.4) for $P(z)$, and obtain the well-known representations for Γ_0 and Γ_1 (see [1, p. 290] and [7, p. 84]).

Lemma 2.2. *$P\in\Gamma_0$ iff P is of the form*

$$P(z)=\frac{1-p}{1-pQ(z)}, \tag{2.5}$$

with $0\leqslant p<1$, and where Q is a p.g.f. with $Q(0)=0$.

Lemma 2.3. *$P\in\Gamma_1$ iff P is of the form*

$$P(z)=\exp[\lambda\{Q(z)-1\}], \tag{2.6}$$

with $\lambda>0$, and where Q is a p.g.f. with $Q(0)=0$.

3. The monotonicity of Γ_α

As a first step we prove that

$$\Gamma_\alpha\subset\Gamma_1 \quad (0\leqslant\alpha<1). \tag{3.1}$$

Theorem 3.1. *If $\{p_n\}\in\Gamma_\alpha$, then $\{p_n\}$ is i.d.*

Proof. From (2.4) it follows that

$$P(z)=\frac{P(\alpha z)}{1-(1-\alpha)zR_\alpha(z)}, \tag{3.2}$$

and iterating this equation we obtain

$$P(z)=\prod_{k=0}^{\infty}\frac{1-\pi_k}{1-\pi_k Q_k(z)},\tag{3.3}$$

with $\pi_k=(1-\alpha)\alpha^k R_\alpha(\alpha^k)$ and $Q_k(z)=zR_\alpha(\alpha^k z)/R_\alpha(\alpha^k)$. It is easily verified that the product in (3.3) is absolutely convergent for $|z|\leq 1$. Therefore, P is the limit of a sequence of products of p.g.f.'s in $\Gamma_0\subset\Gamma_1$ (see (1.7)), and hence $P\in\Gamma_1$.

We state the analogue of Lemmas 2.2 and 2.3 as a corollary.

Corollary 3.2. *Let $0\leq\alpha<1$. Then $P\in\Gamma_\alpha$ iff P is of the form*

$$P(z)=\prod_{k=0}^{\infty}\frac{1-pQ(\alpha^k)}{1-pQ(\alpha^k z)},\tag{3.4}$$

with $0\leq p<1$, and where Q is a p.g.f. with $Q(0)=0$.

We now turn to the general theorem. The proof is fairly simple, but rather hard to find. We shall need the following lemma, which easily follows from Lemma 2.3.

Lemma 3.3. *If $P\in\Gamma_1$, then $P(z)/P(\gamma z)$ is a.m. for $0\leq\gamma\leq 1$. In fact, $P(\gamma)P(z)/P(\gamma z)$ is a p.g.f. in Γ_1.*

Theorem 3.4. $\Gamma_\alpha\subset\Gamma_\beta$ $(0\leq\alpha<\beta\leq 1)$.

Proof. We have to show (cf. Lemma 2.1) that, if $0\leq\alpha<\beta<1$ and if R_α is a.m., then R_β is a.m. To do so we rewrite (2.4) as

$$z(1-\alpha)R_\alpha(z)=1-P(\alpha z)/P(z).\tag{3.5}$$

Now from (3.5) we subtract the same equality with z replaced by βz, and we obtain

$$z(1-\alpha)\{R_\alpha(z)-\beta R_\alpha(\beta z)\}=\frac{P(\alpha\beta z)}{P(\beta z)}-\frac{P(\alpha z)}{P(z)},$$

or, equivalently,

$$\frac{z(1-\alpha)}{P(\alpha z)}\{R_\alpha(z)-\beta R_\alpha(\beta z)\}=\frac{P(\alpha\beta z)}{P(\alpha z)P(\beta z)}-\frac{1}{P(z)},\tag{3.6}$$

the right-hand side of which is symmetric in α and β. So, equating the left-hand side of (3.6) and the same expression with α and β interchanged, we get, after a simple rearrangement,

$$R_\beta(z)-\alpha R_\beta(\alpha z)=\frac{1-\alpha}{1-\beta}\,\frac{P(\beta z)}{P(\alpha z)}\,\{R_\alpha(z)-\beta R_\alpha(\beta z)\}.\tag{3.7}$$

By hypothesis R_α is a.m., and hence $R_\alpha(z)-\beta R_\alpha(\beta z)$, with coefficients $r_n(\alpha)(1-\beta^{n+1})$, is a.m. By Lemma 3.3, and (3.1), $P(\beta z)/P(\alpha z)$ is a.m., and therefore $R_\beta(z)-\alpha R_\beta(\alpha z)$ is a.m., i.e. $r_n(\beta)(1-\alpha^{n+1})\geq 0$ $(n=0,1,2,\ldots)$. It follows that $r_n(\beta)\geq 0$ $(n=0,1,2,\ldots)$, and the theorem is proved.

Looking somewhat more precisely at the identity (3.7) one easily proves the following.

Corollary 3.5. *If for a given distribution R_α is a.m., and if $\alpha < \beta \le 1$, then*

$$r_n(\beta) \ge \frac{1-\alpha}{1-\beta}\,\frac{1-\beta^{n+1}}{1-\alpha^{n+1}}\,r_n(\alpha) \quad (n = 0, 1, 2, \ldots).$$

We restate Theorem 3.4 as a property that generalizes a property of renewal sequences proved in [2].

Corollary 3.6. *If for a given distribution $\{p_n\}$ the quantities $r_n(\alpha)$, for a fixed α $(0 \le \alpha < 1)$ defined by*

$$(1 + \alpha + \ldots + \alpha^n)p_{n+1} = \sum_{k=0}^{n} p_k r_{n-k}(\alpha) \quad (n = 0, 1, 2, \ldots), \tag{3.8}$$

are all non-negative, then also the quantities $r_n(\beta)$ defined by (3.8), *with α replaced by β, are non-negative for all β with $\alpha < \beta \le 1$.*

Goldie's result in [2] is obtained by taking $\alpha = 0$ and $\beta = 1$.

4. Properties of Γ_α

The following theorem generalizes a well-known property of Γ_1.

Theorem 4.1. *For $0 \le \alpha \le 1$ the class Γ_α is closed under weak convergence, i.e. if $P_n \in \Gamma_\alpha$ $(n = 1, 2, \ldots)$, and if $\lim_{n\to\infty} P_n(z) = P(z)$ is a p.g.f., then $P \in \Gamma_\alpha$.*

Proof. For $\alpha = 1$ the theorem is known (cf. [6]). From (2.4) it follows that if $P_n \to P$, then $R_{n,\alpha} \to R_\alpha$ $(0 \le \alpha < 1)$. As absolute monotonicity is preserved under pointwise convergence, R_α is a.m., and hence $P \in \Gamma_\alpha$ by Lemma 2.1.

Next we state some properties of Γ_α, which are well known, or trivial, when all Γ-classes are replaced by Γ_1.

Theorem 4.2. *For $0 \le \alpha < 1$,*

(i) *if $P \in \Gamma_\alpha$, then $P(\gamma z)/P(\gamma) \in \Gamma_\alpha$ $(0 \le \gamma \le 1)$;*

(ii) *if $P \in \Gamma_\alpha$, then $P^\gamma \in \Gamma_\alpha$ $(0 \le \gamma \le 1)$;*

(iii) *if $P \in \Gamma_\alpha$, then*

$$P_{n,\gamma}(z) := \prod_{k=0}^{n-1} P(\gamma^k z)/P(\gamma^k) \in \Gamma_\gamma \quad (\alpha^{1/n} \le \gamma \le 1;\ n = 1, 2, \ldots);$$

(iv) *if $P \in \Gamma_\alpha$, then $P_\gamma(z) := P(\gamma)P(z)/P(\gamma z) \in \Gamma_\alpha$ $(0 \le \gamma \le 1)$;*

(v) *$P \in \Gamma_\alpha$ iff $P_\alpha(z) := P(\alpha)P(z)/P(\alpha z) \in \Gamma_0$.*

Proof. (i) follows directly from Lemma 2.1 and the fact that if $A(z)$ is a.m., then $A(\gamma z)$ is a.m.

(ii) The derivative of $1-\{P(\alpha z)/P(z)\}^{\gamma}$ can be written as

$$\gamma\{P(z)/P(\alpha z)\}^{1-\gamma}\frac{\mathrm{d}}{\mathrm{d}z}\{1-P(\alpha z)/P(z)\}. \tag{4.1}$$

By Lemma 3.3 we know that $P(\alpha)P(z)/P(\alpha z)\in\Gamma_1$ and hence $\{P(\alpha)P(z)/P(\alpha z)\}^{1-\gamma}\in\Gamma_1$, and hence is a.m. The second factor in (4.1) is a.m. by hypothesis, and (ii) follows by Lemma 2.1.

(iii) In order to use Lemma 2.1 again we calculate $1-P_{n,\gamma}(\gamma z)/P_{n,\gamma}(z)=1-P(\gamma^n z)/P(z)$, which is a.m. by hypothesis for $\gamma^n=\alpha$, and hence by Theorem 3.4 for $\gamma^n\geqslant\alpha$.

(iv) We use Lemma 2.1 again, so we have to prove that

$$1-P_\gamma(\alpha z)/P_\gamma(z)=\{P(\gamma z)/P(\alpha\gamma z)\}(1-\alpha)z\{R_\alpha(z)-\gamma R_\alpha(\gamma z)\}$$

is a.m. This follows easily from Lemma 3.3 and the absolute monotonicity of R_α.

(v) We calculate $R_{0,\alpha}$ (i.e. R_0 corresponding to P_α), and get

$$R_{0,\alpha}(z)=(1-\alpha)R_\alpha(z),$$

i.e. $R_{0,\alpha}$ is a.m. iff R_α is a.m. and so (v) follows.

We conclude this section by proving that $\bigcup_{0\leqslant\alpha<1}\Gamma_\alpha$ is dense in Γ_1 in the sense of weak convergence.

Theorem 4.3. *If $P\in\Gamma_1$, then there exists an increasing sequence $\{\alpha_n\}_1^\infty$, with $\alpha_n\to 1$, and a sequence of p.g.f.'s $P_n\in\Gamma_{\alpha_n}$ $(n=1,2,\dots)$, such that*

$$P(z)=\lim_{n\to\infty}P_n(z)\quad(|z|\leqslant 1). \tag{4.2}$$

Proof. We prove the theorem for $0\leqslant z\leqslant 1$; it is then easy to extend the result to $|z|\leqslant 1$. As (4.2) is trivial for $z=1$, we take $z\in[0,1)$. Now, let $P\in\Gamma_1$, then by Lemma 2.3

$$P(z)=\exp[\lambda\{Q(z)-1\}]. \tag{4.3}$$

Take $\alpha_n=1-1/n^2$, and for $n>\lambda$ define P_n by

$$P_n(z)=\prod_{k=0}^{n-1}\frac{1-\lambda/nQ(\alpha_n^k)}{1-\lambda/nQ(\alpha_n^kz)}.$$

By Theorem 4.2 (iii) we know that $P_n\in\Gamma_{\alpha_n}$. We rewrite P_n as

$$P_n(z)=\prod_{k=0}^{n-1}\left[1+\frac{\lambda\{Q(z)-1\}}{n}+\varepsilon_k(n)\right],$$

where it is not difficult (cf. [3] for details) to prove that $\varepsilon_k(n)=o(1/n)$ uniformily in k. It follows that $P_n(z)\to P(z)$ as given by (4.3).

5. Examples

We list a number of examples. The proofs of the following statements are simple, and are therefore omitted (cf. [3]).

1. $\prod_0^{n-1} \frac{1-p\alpha^k}{1-p\alpha^k z} \in \Gamma_\alpha \qquad (0 \leq \alpha \leq 1; n = 1, 2, \ldots).$

2. $\frac{1-p}{1-pz} \frac{1-p\gamma}{1-p\gamma z} \in \Gamma_\alpha$ iff $\alpha \geq \gamma \qquad (0 \leq p < 1; 0 \leq \gamma \leq 1).$

3. $\left(\frac{1-p}{1-pz}\right)^{1+\varepsilon} \notin \Gamma_\alpha \qquad (\varepsilon > 0; 0 < p < 1; 0 \leq \alpha < 1).$

4. $\frac{1-p}{1-pz} \frac{1-p\gamma_1}{1-p\gamma_1 z} \frac{1-p\gamma_1\gamma_2}{1-p\gamma_1\gamma_2 z} \in \Gamma_\alpha \cdot (0 \leq \gamma_1 \leq 1; 0 \leq \gamma_2 \leq 1; \alpha \geq \max(\gamma_1, \gamma_2);$
$0 \leq p < 1).$

5. $\exp\{\lambda(z-1)\} \notin \Gamma_\alpha \qquad (\lambda > 0; 0 \leq \alpha < 1).$

6. If $P(z)$ is such that $R_{\alpha_0}(z) \equiv 1$, then $P \in \Gamma_\alpha$ iff $\alpha \geq \alpha_0$. Distributions of this type have been used in [3] to construct counterexamples to condition (1.8).

7. Let $p_0 = a/(1+a)$ and $p_n = \theta^n/\{(1+a)bn\}$ for $n \geq 1$, with $a > 0$, $b = -\log(1-\theta)$ and $0 < \theta < 1$. Then

$$\{p_n\} \in \Gamma_\alpha \quad \text{iff} \quad ab \geq \frac{2}{1+\alpha} \qquad (0 \leq \alpha \leq 1).$$

This can be proved by induction using (2.3). This example has also been considered in [4] and in [7], where it is noted that p_n is log-convex for $ab \geq 2$.

6. Concluding remarks

A more trivial classification of Γ_1, with properties similar to those of Γ_α, is obtained by considering the classes Λ_u defined for $0 \leq u < \infty$ as follows:

$$P \in \Lambda_u \quad \text{iff} \quad P(z) = \left\{\frac{1-p}{1-pQ(z)}\right\}^u,$$

for some p with $0 \leq p < 1$ and some p.g.f. Q with $Q(0) = 0$.

Most of the properties of the Γ_α can be extended to similar classes of general (i.e. not necessarily lattice) distributions on $[0, \infty)$. Results of this kind will be published elsewhere.

It would be interesting to know whether other choices for $c_n(\alpha)$ exist, satisfying (1.5) and (1.8), especially whether there are such that also satisfy $c_n(\alpha) \sim n^\alpha$ $(n \to \infty)$.

The solutions of the equations (2.3), with given $r_n(\alpha)$, non-negative and satisfying (1.4) or (1.6), may be regarded as generalized renewal sequences. It may be possible to extend some of the properties of renewal sequences proved in [5] to these more general sequences.

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